Financial Mathematics
—Exercises—

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Selected Exercises[1]

1. Prove Itô's isometry.

2. If $dS = \mu S dt + \sigma S dZ$, where S denotes the price of the stock, dZ the increments of a standard Wiener process, and μ, σ, A and n are constants, find the stochastic equations satisfied by
(a) $f(S) = AS$,
(b) $f(S) = S^n$,
(c) Show that $\mathbb{E}[S(T)] = S(t)e^{\mu(T-t)}$.

3. By expanding df in Taylor series to $\mathcal{O}dt$ and using that $(dZ)^2 = dt$, prove that

$$\int_{t_0}^{t} Z(\tau)dZ(\tau) = \frac{1}{2}\left(Z(t)^2 - Z(t_0)^2\right) - \frac{1}{2}(t - t_0).$$

4. Consider the general stochastic differential equation

$$dG = A(G,t)dt + B(G,t)dZ.$$

Use Itô's Lemma to show that it is theoretically possible to find a function $f(G)$ which itself follows a random walk but with zero drift.

5. There are n assets satisfying the following stochastic differential equations

$$dS_i = \mu_i S_i dt + \sigma_i S_i dZ_i \qquad \text{for } i = 1, \ldots, n, .$$

Z_i is a standard Wiener process and

$$dZ_i dZ_j = \rho_{ij} dt$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$.
Derive Itô's Lemma for a function $f(S_1, \ldots, S_n)$ of the n assets $S_1, \ldots, S_n$.

6. Find the most general solution of the PDE

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \tag{1}$$

that has the special form
(a) $V = V(S)$;
(b) $V = A(t)B(S)$.

[1]Selected from *The Mathematics of Financial Derivatives —A Student Introduction*, by Paul Wilmott, Sam Howison and Jeff Dewynne.

7. Let S satisfy

$$dS = \alpha\left(\mu - \ln S\right) S dt + \sigma S dW\,, \tag{2}$$

where α and σ are non-negative constants, μ is a constant, and W a standard brownian motion.
Show that

$$x_T = x_t e^{-b(T-t)} + \frac{a}{b}\left(1 - e^{-b(T-t)}\right) + \sigma e^{-bT}\int_t^T e^{bs} dW\,,$$

where $x_t = \ln S_t$, $a = \alpha\hat{\mu}$ and $b = \alpha$.

8. Use the above to calculate $\mathbb{E}_t[S_T]$.

9. **European Call Option as a Consumer Hedge** Suppose a consumer of heating oil, with a physical supply contract based upon floating market prices, is concerned about market prices rising but wishes to benefit should market prices drop. They decide to enter into an OTC (over-the-counter) European call option on heating oil futures prices with a strike price of 2.00$ per gallon for which they pay a premium of 0.30$ per gallon.

   - Draw a payoff diagram in the absence of the call option
   - Draw the payoff of the call
   - Draw the payoff diagram of the exposure of the energy consumer in the presence of the European call

10. **Crude producer Hedge Using European Put Options** Suppose a producer of brent crude oil is concerned about falling market prices, but does not want to sacrifice its upside if prices should rise. It sells its crude oil on a floating market index, and it decides to enter into a European put option on Brent crude oil to indemnify itself against price drops. It enters into a put option contract with strike of 110.00$ per barrel and pays 2.00$ per barrel for the protection.

    - Draw a payoff diagram in the absence of the put option
    - Draw the payoff of the put
    - Draw the payoff diagram of the exposure of the energy producer in the presence of the European put

11. Assume spot prices follow, $\ln S_t = g(t) + \chi_t + \xi_t$ under the physical measure, where

$$\begin{aligned} d\chi_t &= -\kappa\chi_t dt + \sigma_\chi dW_\chi, \\ d\xi_t &= \mu_\xi dt + \sigma_\xi dW_\xi, \end{aligned}$$

with $dW_\chi dW_\xi = \rho dt$.
Show that

$$\begin{aligned} F(t,T) &= \exp\left(g(T) + e^{-\kappa^*(T-t)}\chi_t + \xi_t + A(T-t)\right), \qquad (3) \\ A(T-t) &= \mu_\xi^*(T-t) \\ &\quad - \left(1 - e^{-\kappa(T-t)}\right)\frac{\alpha}{\kappa^*} \\ &\quad + \frac{1}{2}\left(\left(1 - e^{-2\kappa^*(T-t)}\right)\frac{\sigma_\chi^2}{2\kappa^*}\right) \\ &\quad + \frac{1}{2}\left(\sigma_\xi^2(T-t) + 2\left(1 - e^{-\kappa^*(T-t)}\right)\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa^*}\right). \end{aligned}$$

12. Assume

$$\begin{aligned} dX_t &= \left(\mu - \delta_t - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dW_1, \qquad (4) \\ d\delta_t &= \kappa(\alpha - \delta_t)dt + \sigma_2 dW_2, \qquad (5) \end{aligned}$$

with W_1 and W_2 correlated Wiener processes. Here X_t modelled the spot price with drift μ and volatility σ_1, δ_t modelled the stochastic convenience yield, an OU process with mean reversion rate, κ, mean reversion level α and volatility σ_2.

Discuss the connection between the model (4), (5) and the LT/ST presented here. How is the stochastic convenience yield connected to the short-term shocks.

13. Let

$$d(\ln S) = \left(\alpha - \frac{1}{2}\sigma^2 - \lambda k\right)dt + \sigma dW + \ln Y dN.$$

where α, σ, k are constants, W is a standard Brownian motion, N is a counting process with intensity λ, and Y are iid. W, N, Y are independent.

Assume that $\alpha = r$ and $k = \mathbb{E}_Y[Y] - 1$. Show that

$$\mathbb{E}\left[e^{-rT}\max(S_T - K, 0)\right] = \sum_{n=0}^{\infty}\left[\frac{e^{-\lambda T}(\lambda T)^n}{n!}\mathbb{E}_Y\left[C^E(S_0 Y^n e^{-\lambda(\mathbb{E}_Y[Y]-1)T}, 0; T, K)\right]\right],$$

where C^E denotes the value of a European call option in the Black-Scholes model. Moreover, show that

$$\mathbb{E}\left[e^{-rT}\max(S_T - K, 0)\right] \geq C^E(S_0, 0; T, K).$$

Explain when do we get strict equality in the above equation.

14. Assume

$$\ln S_t = g(t) + Y_t,$$

where S_t is the spot price, $g(t)$ is a deterministic seasonality function and

$$dY_t = -\alpha Y_t dt + \sigma(t) dZ + \ln J dq_t. \tag{6}$$

where α is the speed of mean reversion, $\sigma(t)$ is the time dependent volatility, J is the proportional random jump size, dZ_t is the increment of the standard Brownian motion and q_t is a homogeneous Poisson process with intensity l. Moreover, J, q_t and Z_t are independent.

(a) Use Ito's lemma to find the expression for dS/S.

(b) Show that

$$\begin{aligned} x_T = \quad & g(T) + (x_t - g(t))\, e^{-\alpha(T-t)} - \lambda\left(1 - e^{-\alpha(T-t)}\right) \\ & + \int_t^T \sigma(s) e^{-\alpha(T-s)} d\hat{Z}_s + \int_t^T e^{-\alpha(T-s)} \ln J dq_s \end{aligned} \tag{7}$$

where $x_t = \ln S_t$.

(c) Find the forward price $F(t,T) = \mathbb{E}_t[S_T | \mathcal{F}_t]$.

(d) Discuss (in words) the assumption of no jump-risk in the model.

15. For the problem set up, please see BESSEMBINDER and LEMMON's paper which is attached in the moodle.

(a) Show

$$P_W = a\left(\frac{Q^D}{N_P}\right)^{c-1}, \tag{8}$$

which is the equilibrium spot price.

(b) Show that

$$\mathbb{C}\left[\rho_{Pi}, P_W\right] = \frac{1}{a^x}\mathbb{C}\left[P_W^{x+1}, P_W\right] - \frac{1}{ca^x}\mathbb{C}\left[P_W^{x+1}, P_W\right] \tag{9}$$

and

$$\mathbb{C}\left[\rho_{Rj}, P_W\right] = P_R \mathbb{C}\left[Q_{Rj}, P_W\right] - \mathbb{C}\left[P_W Q_{Rj}, P_W\right]. \tag{10}$$

(c) Show that the equilibrium forward price

$$P_F = \mathbb{E}[P_W] - \frac{N_P}{Nca^x}\left[cP_R \mathbb{C}\left[P_W^x, P_W\right] - \mathbb{C}\left[P_W^{x+1}, P_W\right]\right], \tag{11}$$

where $N = (N_R + N_P)/A$ is a measure that reflects the number of firms in the industry and the degree to which they are concerned with risk.

(d) Using a Taylor's series to show that

$$P_F = \mathbb{E}[P_W] + \alpha\mathbb{V}[P_W] + \gamma\mathbb{S}[P_W] \tag{12}$$

where

$$\alpha = \frac{N_P(x+1)}{Nca^x}\left([\mathbb{E}[P_W]]^x - P_R[\mathbb{E}[P_W]]^{x-1}\right)$$

and

$$\gamma = \frac{N_P(x+1)}{2Nca^x}\left(x[\mathbb{E}[P_W]]^{x-1} - (x-1)P_R[\mathbb{E}[P_W]]^{x-2}\right).$$

(e) Let the $P_F - \mathbb{E}[P_W]$ be the forward premium, prove that this value is decreasing in the variance of the spot price.

(f) Explain why in the case of zero skewness it is the case that power retailers profits are positively exposed to increases in P_W.

16. (a) What is the value of an option with payoff $\mathcal{H}(S-K)$?
(b) What is the value of an option with payoff $\frac{1}{d}(\mathcal{H}(S-K) - \mathcal{H}(S-K-d))$?

17. The European asset-or-nothing call pays S if $S > K$ at expiry, and nothing if $S \leq K$. What is its value?

18. What is the probability that a European call will expire in-the-money?

19. An option has a general payoff $\Lambda(S)$ at time T, and its value is $V(S,t)$. Show how to synthesise it from vanilla call options with varying exercise prices; that is, how to find the 'density' $f(K)$ of calls, with the same expiry T, exercise price K and price $C(S,t;K)$, such that

$$V(S,t) = \int_0^\infty f(K)C(S,t;K)dK.$$

Verify that your answer is correct
(a) when $\Lambda(S) = \max(S-K,0)$;
(b) when $\Lambda(S) = S$. (What is the synthesizing portfolio here?)

20. What is the random walk followed by the futures price F in the Black-Scholes model?

21. The instalment option has the same payoff as a vanilla call or put option; it may be European or American. Its unusual feature is that, as well as paying the initial premium, the holder must pay 'instalments' during the life of the option. The instalments may be paid either continuously or discretely. The holder can choose at any time to stop paying the instalments, at which point the contract is cancelled and the option ceases to exist. When instalments are paid continuously at a rate $L(t)$ per unit time, derive the differential equation satisfied by the option price. What new constraint must it satisfy?

22. Show that the value of European call option on an asset that pays a constant continuous dividend yield lies below the payoff for large enough values of S. Show also that the call on an asset with dividends is less valuable than the call on an asset without dividends.

23. Consider American vanilla call and put options, with prices C and P. Derive the following inequalities (the second part of the last inequality is the version of put-call parity result appropriate for American options):

$$P \geq \max(K - S, 0), \qquad C \geq S - Ke^{-r(T-t)},$$
$$S - K \leq C - P \leq S - Ke^{-r(T-t)}.$$

Also show that, without dividends, it is never optimal to exercise an American call option.

24. Consider the Black Scholes PDE

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \tag{13}$$

where $V(S,t)$ is the value of a European-style option written on the underlying stock S with expiry T, strike K, r is the risk-free rate and $\sigma > 0$.

(a) Make the substitution $z = \ln S$ in (13) and show that the Black-Scholes PDE becomes

$$V_t + \frac{1}{2}\sigma^2 V_{zz} + \left(r - \frac{1}{2}\sigma^2\right) V_z - rV = 0.$$

Straightforward algebra delivers the result.

(b) Show that applying the Fourier transform to the Black-Scholes PDE we obtain the ODE

$$-\frac{\partial \hat{V}}{\partial t} = \mathcal{L}\hat{V},$$

for a certain function $\mathcal{L}(\xi)$ and the solution to the ODE is given by:

$$\hat{V}(\xi,t) = \hat{V}(\xi,T)e^{(T-t)\mathcal{L}(\xi)},$$

where $\hat{V}(\xi,t)$ is the FT of $V(z,t)$.
[The FT of a function $g(x)$ is given by

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} g(x) dx,$$

with inverse transform

$$g(x) = \frac{1}{2\pi} \int_{i\xi_i - \infty}^{i\xi_i + \infty} e^{-i\xi x} \hat{g}(\xi) d\xi,$$

where $\xi_i = \text{Im } \xi$.]

25. Merton:

The SDE in Merton's model can be written as

$$dS = \tilde{\mu} S dt + \sigma S dW + S(J-1)dq,$$

where dW is the increment of Brownian motion, q is a Poisson process with intensity λ, J is a random variable (W, q, J are independent) and $\tilde{\mu} = \mu - \lambda k$ where μ, and k are constants. Moreover, $dq = 0$ with probability $1 - \lambda dt$ and $dq = 1$ with probability λdt.

(a) Defining the return as $x = \frac{dS}{S}$ show that

$$\mathbb{E}[x] = \mu dt \quad \text{if and only if} \quad k = \mathbb{E}[J-1].$$

(b) By integrating the SDE between 0 and t and using Itô's Lemma prove that

$$S_t = S_0 e^{\hat{\mu} t + \sigma W_t} J(n),$$

where $\hat{\mu} = \mu - \lambda k - \frac{\sigma^2}{2}$ and

$$J(n) = \begin{cases} 1 & \text{if } n = 0\,, \\ \prod_{i=1}^{n} J_i & \text{if } n \geq 1\,, \end{cases}$$

where the J_i are independently and identically distributed and n is Poisson distributed with parameter λt.

(c) In order to obtain a PDE for this problem you must assume that $\mathbb{E}[d\Pi] = r\Pi dt$. Under which specific assumptions (relevant to this problem) might one assume this?

(d) Based on the previous assumption and on a Δ-hedging strategy, derive the pricing PDE for Merton's problem

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda k) S \frac{\partial V}{\partial S} - rV + \lambda \mathbb{E}[V(SJ,t) - V(S,t)] = 0.$$